

# DtN isospectrality, flat metrics with non-trivial holonomy and comparison formulas for determinants of Laplacian

Luc Hillairet\*, Alexey Kokotov†

November 17, 2015

MAPMO (UMR 7349 Université d'Orléans-CNRS) UFR Sciences, Bâtiment de  
mathématiques rue de Chartres, BP 6759 45067 Orléans Cedex 02

Department of Mathematics and Statistics, Concordia University, 1455 de  
Maisonnette Blvd. West, Montreal, Quebec, H3G 1M8 Canada

**Abstract.** We study comparison formulas for  $\zeta$ -regularized determinants of self-adjoint extensions of the Laplacian on flat conical surfaces of genus  $g \geq 2$ . The cases of trivial and non-trivial holonomy of the metric turn out to differ significantly.

## 1 Introduction

Let  $X$  be a compact Riemann surface of genus  $g \geq 2$  and let  $D = P_1 + \dots + P_{2g-2}$  be a positive divisor of degree  $2g - 2$  on  $X$ . For simplicity, we consider only the generic situation: all the points  $P_k$  are assumed to be distinct. Then, due to the Troyanov theorem ([9]), there exists a unique (up to rescaling) conformal metric  $\mathbf{m}$  on  $X$  that is flat with conical singularities of angle  $4\pi$  at  $P_1, \dots, P_{2g-2}$ . If the divisor  $D$  belongs to the canonical class then there also exists a holomorphic one form  $\omega$  on  $X$  with simple zeros at  $P_1, \dots, P_{2g-2}$ . In that situation we thus obtain

$$\mathbf{m} = |\omega|^2, \quad (1.1)$$

so that the Troyanov metric  $\mathbf{m}$  has trivial holonomy: i. e. the parallel transport along any loop that avoids the conical singularities of  $X$ , maps any tangent vector to  $X$  to itself. Indeed, after the parallel transport along such a loop, a tangent vector has turned

---

\*E-mail: luc.hillairet@univ-orleans.fr

†E-mail: alexey.kokotov@concordia.ca

for an angle  $2\pi k$  with  $k \in \mathbb{Z}$ . On the other hand, any flat metric with conical singularities that has trivial holonomy is of the form (1.1). It follows that the Troyanov metric  $\mathbf{m}$  corresponding to the divisor  $D$  and conical angles  $4\pi$  has trivial holonomy if and only if the divisor  $D$  belongs to the canonical class.

Consider now a general flat metric  $\mathbf{m}$  with conical singularities of angle  $4\pi$ , and define the Laplace operator  $\Delta^{\mathbf{m}} : L^2(X, \mathbf{m}) \rightarrow L^2(X, \mathbf{m})$  that is associated with it. This operator is initially defined on the space  $C_0^\infty(X \setminus \{P_1, \dots, P_{2g-2}\})$  of smooth functions that vanish in a neighbourhood of the conical points. The closure of  $\Delta^{\mathbf{m}}$  is a symmetric operator in  $L^2(X, \mathbf{m})$  with deficiency indices  $(6g - 6, 6g - 6)$ . It thus has infinitely many self-adjoint extensions. Among these, the Friedrichs extension,  $\Delta_F$ , is extensively studied. Explicit formulas for the (modified: i. e. with zero modes excluded)  $\zeta$ -regularized determinant  $\det^* \Delta_F$  are found in [8] (in the case of trivial holonomy) and [7] (in the general case).

The domain of the Friedrichs extension may include only functions that are bounded near the conical singularities. As a consequence, when the metric  $\mathbf{m}$  has trivial holonomy, the function

$$u = \frac{\psi}{\omega}, \quad (1.2)$$

where  $\omega$  is the holomorphic one form that defines the metric  $\mathbf{m} = |\omega|^2$  and  $\psi$  is an arbitrary holomorphic one form on  $X$  not proportional to  $\omega$ , belongs to  $L^2(X, \mathbf{m})$ , satisfies the equation  $\Delta^{\mathbf{m}} u = 0$  in the classical sense outside the singularities  $P_1, \dots, P_{2g-2}$  but *does not* belong to the domain of the  $\Delta_F$ . It turns out that there exists another natural s. a. extension of  $\Delta^{\mathbf{m}}$  that we denote by  $\Delta_{\text{hol}}$  whose domain does contain the functions of the form (1.2). Choosing a s.a. extensions is equivalent to prescribe a certain kind of asymptotic behaviour near the conical points. For instance, functions in  $\text{dom}(\Delta_F)$  have to be bounded but can have both holomorphic and antiholomorphic terms in the asymptotic expansion near the conical points. On the contrary the functions in  $\text{dom}(\Delta_{\text{hol}})$  are not necessarily bounded but may only have holomorphic terms in this expansion (hence the name; see section 2.1).

In [5] it was found a general comparison formula relating  $\det^* \Delta_F$  to the determinant of any (regular) s. a. extension of the Laplacian for any flat conformal conical metric on  $X$  (with arbitrary conical angles). The main player in this comparison formula is the so-called  $S$ -matrix,  $S(\lambda)$ , of the conical surface  $X$ . The goal of this paper is twofold. First we will apply the results of [5] to relate the modified zeta-regularized determinant  $\det^*(\Delta_{\text{hol}})$  with the determinant of the Friedrichs extension. Then we will study in full detail the  $S$ -matrix in the special case of metrics with conical angles  $4\pi$ : we will find explicit expressions for the entries of  $S(\lambda)$  and their derivatives at  $\lambda = 0$  through invariant holomorphic objects related to the Riemann surface  $X$ . This will result in a nice explicit expression for the determinant of the holomorphic s.a. extension. The following theorem illustrates our result in genus 2.

**Theorem 1.** *Let  $X$  be a Riemann surface of genus 2. For any two points  $P_1$  and  $P_2$  on  $X$ , let  $\mathbf{m}$  be the Troyanov flat metric with conical singularities of angle  $4\pi$  at  $P_1$  and  $P_2$  and let  $\Delta_F$  and  $\Delta_{\text{hol}}$  be the corresponding s.a. extensions of  $\Delta^{\mathbf{m}}$ .*

*If the divisor  $P_1 + P_2$  is not in the canonical class or, equivalently, if  $\mathbf{m}$  does not have trivial holonomy, then*

$$\det^* \Delta_{\text{hol}} = c_2 \begin{vmatrix} B(P_1, P_1) & B(P_1, P_2) \\ B(P_2, P_1) & B(P_2, P_2) \end{vmatrix} \det^* \Delta_F, \quad (1.3)$$

where  $B(P_i, P_j)$  is the Bergman reproducing kernel for the holomorphic differentials on  $X$  which is evaluated at  $P_i, P_j$  using distinguished holomorphic local parameters. The constant  $c_2$  is universal.

If the divisor  $P_1 + P_2$  is in the canonical class or, equivalently, if  $\mathbf{m}$  has trivial holonomy, then

$$\det^*(\Delta_{\text{hol}}) = \det^*(\Delta_F). \quad (1.4)$$

In the trivial holonomy case, it was previously observed by the first author ([4]) that the operators  $\Delta_F$  and  $\Delta_{\text{hol}}$  are DtN isospectral (see the precise definition below). This immediately implies that their  $\zeta$ -regularized determinants coincide.

It should be noted that when the divisor  $P_1 + P_2$  enters the canonical class the matrix

$$\begin{pmatrix} B(P_1, P_1) & B(P_1, P_2) \\ B(P_2, P_1) & B(P_2, P_2) \end{pmatrix}$$

becomes degenerate and the dimension of the kernel of  $\Delta_{\text{hol}}$  gets larger (by one unit).

Relation (1.4) together with the comparison formula from [5] and the explicit information about the entries of  $S(\lambda)$  obtained in the present paper, lead to identity (4.6) below, which presents the second main result of the paper. The latter identity seems to be rather nontrivial and we failed to find any direct way to prove it.

We put a certain effort to make it possible to read this paper independently of [5] (which is more technically involved): in our case of conical singularities of angle  $4\pi$  the results and notations from [5] can be significantly simplified, we will briefly remind the reader all the constructions and main ideas from [5] using this opportunity to make the presentation more transparent.

**Acknowledgements.** We thank D. Korotkin for numerous discussions. We are extremely grateful to K. Shramov and N. Tyurin for their help with the proof of Proposition 2; the second author also acknowledges useful discussions with P. Zograf, in particular, the question about the relation between  $\det \Delta_F$  and  $\det \Delta_{\text{hol}}$  was raised by P. Zograf in one of these discussions.

The research of AK was supported by NSERC.

## 2 $S$ -matrix of Euclidean surface with conical singularities

In this section we briefly remind the setting of paper [5] adopting it to our (less general) situation.

### 2.1 Self-adjoint extensions of Laplacian

Let  $X$  be a Riemann surface of genus  $g \geq 2$  and let  $\mathbf{m}$  be a flat conformal metric on  $X$  with conical singularities  $P_1, \dots, P_{2g-2}$  of conical angles  $4\pi$ . Let  $\Delta$  be the closure of the Laplace operator (corresponding to  $\mathbf{m}$ ) with domain

$$C_0^\infty(X \setminus \{P_1, \dots, P_{2g-2}\}) \subset L_2(X, \mathbf{m})$$

and let  $\Delta^*$  be its adjoint operator. Let  $\xi_k$  be the distinguished holomorphic local parameter in a vicinity of  $P_k$ , i . e.

$$\mathbf{m} = 4|\xi_k|^2 |d\xi_k|^2$$

near  $P_k$ . Near any  $P_k$ , any  $u \in \text{dom}(\Delta^*)$  has an asymptotic expansion

$$\begin{aligned} u(\xi_k, \bar{\xi}_k) &= a_k(u) \frac{i}{\sqrt{2\pi}} \log |\xi_k| + b_k(u) \frac{1}{2\sqrt{\pi}} \frac{1}{\xi_k} + c_k(u) \frac{1}{2\sqrt{\pi}} \frac{1}{\bar{\xi}_k} \\ &\quad + \frac{i}{\sqrt{2\pi}} d_k(u) + \frac{1}{2\sqrt{\pi}} e_k(u) \xi_k + \frac{1}{2\sqrt{\pi}} f_k(u) \bar{\xi}_k \\ &\quad + o(|\xi|). \end{aligned} \quad (2.1)$$

the factors  $\frac{1}{2\sqrt{\pi}}$ , etc are introduced in order to get the standard Darboux basis (see formula (2.2) below).

Let  $u, v \in \mathcal{D}(\Delta^*)$ , then Green formula implies the relation

$$\begin{aligned} \Omega([u], [v]) &:= \langle \Delta^* u, \bar{v} \rangle - \langle u, \Delta^* \bar{v} \rangle \\ &= \sum_{k=1}^{2g-2} X_k(u) \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix} X_k(v)^t, \end{aligned} \quad (2.2)$$

where  $X_k(u) = (a_k(u), b_k(u), \dots, f_k(u))$  and  $\Omega$  is the symplectic form on the factor space  $\text{dom}(\Delta^*)/\text{dom}(\Delta)$ .

The self-adjoint extensions of  $\Delta$  are in one to one correspondence with the Lagrangian subspaces of  $\text{dom}(\Delta^*)/\text{dom}(\Delta)$ ; the Friedrichs extension corresponds to the lagrangian subspace

$$a_k(u) = b_k(u) = c_k(u) = 0, \quad k = 1, \dots, 2g - 2,$$

the holomorphic extension corresponds to the lagrangian subspace

$$a_k(u) = c_k(u) = f_k(u) = 0, \quad k = 1, \dots, 2g - 2.$$

## 2.2 Special solutions and $S$ -matrix

For any  $s \in \mathbb{R}$ , denote by  $H_F^s(X) := \text{dom}(\Delta_F^{\frac{s}{2}})$ . Since  $X$  is compact, the spectrum of  $\Delta_F$  is discrete and consists only in finite multiplicity eigenvalues. Let  $(\phi_\ell)_{\ell \geq 0}$  be a real orthonormal basis of eigenfunctions for  $\Delta_F$  and denote by  $\lambda_\ell$  the eigenvalue that corresponds to  $\phi_\ell$ . This basis will be fixed throughout the paper.

We define the linear functionals  $\Lambda_{\xi_k}, \Lambda_{\bar{\xi}_k}$  and  $\Lambda_{0k}$  on  $H_F^2(X)$  in such a way that any  $u \in H_F^2(X) (= \text{dom}(\Delta_F))$  has the following asymptotic expansion near  $P_k$  :

$$u(\xi_k, \bar{\xi}_k) = \frac{i}{\sqrt{\pi}} \Lambda_{0k}(u) + \frac{1}{2\sqrt{\pi}} \Lambda_{\xi_k}(u) \cdot \xi_k + \frac{1}{2\sqrt{\pi}} \Lambda_{\bar{\xi}_k}(u) \cdot \bar{\xi}_k + o(|\xi_k|).$$

Since  $\Lambda_{\xi_k}$  are (continuous) linear functionals on  $H_F^2(X)$  it follows that the sequence  $\left( \frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell - \lambda} \right)_{\ell \geq 0}$  is in  $\ell^2(\mathbb{N}, \mathbb{C})$ .

Let  $\lambda$  do not belong to the spectrum  $\Delta_F$ . We define  $G_{\xi_k}$  by

$$G_{\xi_k} := (\Delta_F - \lambda)^{-1} \Lambda_{\xi_k}$$

and  $G_{\bar{\xi}_k}, G_{0k}$  by a similar formula.

By definition, we have

$$\forall k, \quad G_{\xi_k}(\cdot, \lambda) = \sum_{\ell \geq 0} \frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell - \lambda} \phi_\ell. \quad (2.3)$$

There are similar expressions for  $G_{\bar{\xi}_k}$ .

We recall that  $G_{\xi_k}$  can be constructed in the following way. Start with  $F_{\xi_k} := \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\xi_k} \chi(\xi_k, \bar{\xi}_k)$  where  $\chi$  is a cutoff function that is identically one near  $P_k$ . We then have

$$G_{\xi_k} = F_{\xi_k} - (\Delta_F - \lambda)^{-1}(\Delta^* - \lambda)F_{\xi_k}.$$

Thus, by construction,  $G_{\xi_k} - F_{\xi_k}$  belongs to  $H_F^2(X)$  and so does  $\partial_\lambda G_{\xi_k}$ . The latter has the following (convergent in  $L^2(X)$ ) expression in the chosen orthonormal basis:

$$\partial_\lambda G_{\xi_k}(\cdot, \lambda) = \sum_{\ell \geq 0} \frac{\Lambda_{\xi_k}(\phi_\ell)}{(\lambda_\ell - \lambda)^2} \phi_\ell. \quad (2.4)$$

By evaluating the linear functionals  $\Lambda$  on  $G_{\xi_k} - F_{\xi_k}$ , we obtain a  $(6g - 6) \times (6g - 6)$  matrix  $S(\lambda)$ .

In what follows we will often use the following convenient and self-explanatory notation: the entries of  $S$  will be denoted by expressions of the type  $S^{\bar{\xi}_k \xi_j}(\lambda)$  that will correspond to the coefficient of  $\xi_j$  in the expansion of  $G_{\bar{\xi}_k}$ . Thus, by definition, near  $P_k$  we will have

$$G_{\xi_k} = \frac{1}{2\sqrt{\pi}} \frac{1}{\xi_k} + \frac{i}{\sqrt{2\pi}} S^{\xi_k 0_k}(\lambda) + \frac{1}{2\sqrt{\pi}} S^{\xi_k \xi_k}(\lambda) \xi_k + \frac{1}{2\sqrt{\pi}} S^{\xi_k \bar{\xi}_k}(\lambda) \bar{\xi}_k + o(|\xi_k|)$$

as  $\xi_k \rightarrow 0$  and near  $P_j$  we will have

$$G_{\xi_k} = \frac{i}{\sqrt{2\pi}} S^{\xi_k 0_j}(\lambda) + \frac{1}{2\sqrt{\pi}} S^{\xi_k \xi_j}(\lambda) \xi_j + \frac{1}{2\sqrt{\pi}} S^{\xi_k \bar{\xi}_j}(\lambda) \bar{\xi}_j + o(|\xi_j|)$$

as  $\xi_j \rightarrow 0$ .

It is convenient to represent this  $S$ -matrix with a block structure

$$S(\lambda) = \begin{pmatrix} S_{00}(\lambda) & S_{0a}(\lambda) & S_{0h}(\lambda) \\ S_{a0}(\lambda) & S_{aa}(\lambda) & S_{ah}(\lambda) \\ S_{h0}(\lambda) & S_{ha}(\lambda) & S_{hh}(\lambda) \end{pmatrix},$$

where the indices  $a$  and  $h$  stand for holomorphic and anti-holomorphic. Thus, for instance, the  $S_{ah}$  block records the coefficients of the anti-holomorphic part of the  $G_{\xi_k}$ :

$$S_{ah}^{jk} = S^{\bar{\xi}_j \xi_k}$$

The  $S$ -matrix is holomorphic outside the spectrum of  $\Delta_F$ . The blocks  $S_{00}$ ,  $S_{0a}$ ,  $S_{0h}$ ,  $S_{h0}$  and  $S_{a0}$  blow up as  $\lambda \rightarrow 0$ , whereas the blocks  $S_{aa}$ ,  $S_{ah}$ ,  $S_{ha}$  and  $S_{hh}$  are regular at  $\lambda = 0$ .

In the next subsection we find explicit expressions for  $S_{aa}(0)$ ,  $S_{ah}(0)$ ,  $S_{ha}(0)$  and  $S_{hh}(0)$ . These expressions present the first new result of this paper.

## 2.3 $S(0)$ through Schiffer and Bergman kernels

### 2.3.1 Some kernels from complex geometry

Chose a marking for the Riemann surface  $X$ , i.e. a canonical basis  $a_1, b_1, \dots, a_g, b_g$  of  $H_1(X, \mathbf{Z})$ . Let  $\{v_1, \dots, v_g\}$  be the basis of holomorphic differentials on  $X$  that is normalized via

$$\int_{a_i} v_j = \delta_{ij}.$$

Then the matrix of  $b$ -periods of the marked Riemann surface  $X$  is defined as

$$\mathbb{B} = [b_{ij}]_{i,j \leq g}, \quad \text{with } b_{ij} := \int_{b_i} v_j.$$

Let  $W(\cdot, \cdot)$  be the canonical meromorphic bidifferential on  $X \times X$ , (see [3]). Recall that it is symmetric ( $W(P, Q) = W(Q, P)$ ), and normalized by

$$\int_{a_i} W(\cdot, P) = 0.$$

Recall also the following identities:

$$\int_{b_j} W(\cdot, P) = 2\pi i v_j(P).$$

The only pole of the bidifferential  $W$  is a double pole along the diagonal  $P = Q$ . In any holomorphic local parameter  $x(P)$ , the following asymptotic expansion near the diagonal holds:

$$W(x(P), x(Q)) = \left( \frac{1}{(x(P) - x(Q))^2} + H(x(P), x(Q)) \right) dx(P) dx(Q), \quad (2.5)$$

where

$$H(x(P), x(Q)) = \frac{1}{6} S_B(x(P)) + O(x(P) - x(Q)),$$

as  $Q \rightarrow P$ , with  $S_B(\cdot)$  the Bergman projective connection.

The Schiffer bidifferential (or Schiffer kernel) is defined by

$$\mathcal{S}(P, Q) = W(P, Q) - \pi \sum_{i,j} (\Im \mathbb{B})_{ij}^{-1} v_i(P) v_j(Q),$$

and the Schiffer projective connection,  $S_{\text{Sch}}$ , by the asymptotic expansion

$$\mathcal{S}(x(P), x(Q)) = \left( \frac{1}{(x(P) - x(Q))^2} + \frac{1}{6} S_{\text{Sch}}(x(P)) + O(x(P) - x(Q)) \right) dx(P) dx(Q)$$

It follows the equality

$$S_{\text{Sch}}(x) = S_B(x) - 6\pi \sum_{i,j} (\Im \mathbf{B})_{ij}^{-1} v_i(x) v_j(x). \quad (2.6)$$

In contrast to the canonical meromorphic differential and the Bergman projective connection, the Schiffer bidifferential and the Schiffer projective connection are independent of the marking of the Riemann surface  $X$ .

We will also need the Bergman kernel which is the reproducing kernel for holomorphic differentials on  $X$ . Its expression is given by

$$B(x, \bar{x}) = \sum_{i,j} (\Im \mathbb{B})_{ij}^{-1} v_i(x) \overline{v_j(x)}. \quad (2.7)$$

### 2.3.2 Harmonic functions on $X$

Let  $X$  be a compact Riemann surface of genus  $g > 1$  provided with flat conformal metric  $m$  with  $2g - 2$  conical singularities at  $P_1, \dots, P_{2g-2}$  of conical angles  $4\pi$ . We do not assume that this metric has trivial holonomy: the divisor class  $P_1 + \dots + P_{2g-2}$  is not necessarily canonical.

The harmonic functions on  $X \setminus \{P_k\}$  which are square-integrable are uniquely determined by their singular behaviour near the points  $P_k$ . Thus, there exists a unique harmonic function  $H_k$  with the asymptotic behavior

$$H_k(\xi_k, \bar{\xi}_k) = \frac{1}{\xi_k} + a_{kk} + b_{kk}\xi_k + c_{kk}\bar{\xi}_k + o(|\xi_k|) \quad \text{at } P_k$$

$$H_k(\xi_j, \bar{\xi}_j) = a_{kj} + b_{kj}\xi_j + c_{kj}\bar{\xi}_j + o(|\xi_j|) \quad \text{at } P_j; \quad j \neq k$$

(we remind the reader that  $\xi_k$  denotes the distinguished local parameter for  $\mathbf{m}$  near  $P_k$ ). Comparing with the definition of  $G_{\xi_k}$  we see that

$$H_k = 2\sqrt{\pi}G_{\xi_k},$$

and thus

$$\begin{aligned} b_{kj} &= S^{\xi_k \xi_j}(0) \\ c_{kj} &= S^{\xi_k \bar{\xi}_j}(0). \end{aligned}$$

Notice that  $S^{\bar{\xi}_k \bar{\xi}_j}(0) = \overline{S^{\xi_k \xi_j}(0)}$  and  $S^{\bar{\xi}_k \xi_j}(0) = \overline{S^{\xi_k \bar{\xi}_j}(0)}$  and, therefore, the  $S(0)$  is completely known once we find  $b_{kj}$  and  $c_{kj}$ .

**Proposition 1.** *The following relations hold*

$$\begin{aligned} S^{\xi_k \xi_k} &= -\frac{1}{6} S_{\text{Sch}}(\xi_k) \Big|_{\xi_k=0} \\ S^{\xi_k \xi_j} &= -\mathcal{S}(P_k, P_j), \quad k \neq j \\ S^{\xi_k \bar{\xi}_j} &= -\pi B(P_k, P_j), \end{aligned}$$

where the values of (bi)differentials at the points  $P_i$  are taken w. r. t. the distinguished local parameter  $\xi_i$ .

*Proof.* Introduce the one forms  $\Omega_k$  and  $\Sigma_k$  on  $X$ :

$$\begin{aligned} \Omega_k &= -W(\cdot, P_k) + 2\pi i \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} \{\Im v_\beta(P_k)\} v_\alpha(\cdot) \\ \Sigma_k &= -iW(\cdot, P_k) + 2\pi i \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} \{\Re v_\beta(P_k)\} v_\alpha(\cdot), \end{aligned}$$

where

$$v_\beta(P_k) := v_\beta(\xi_k)|_{\xi_k=0}.$$

All the periods of the differentials  $\Omega_k$  and  $\Sigma_k$  are pure imaginary, therefore, the following expression correctly defines a function  $H_k$  on  $X$

$$H_k(Q) = \Re \left\{ \int_{P_0}^Q \Omega_k \right\} - i \Re \left\{ \int_{P_0}^Q \Sigma_k \right\} \quad (2.8)$$

where  $P_0 \neq P_k$  is an arbitrary base point.

Simple calculation shows that

$$H_k(\xi_k, \bar{\xi}_k) = \frac{1}{\xi_k} + \text{const} + \left[ -\frac{1}{6} S_B(P_k) + \pi \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} v_\alpha(P_k) v_\beta(P_k) \right] \xi_k +$$

$$\left[ -\pi \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} \bar{v}_\alpha(P_k) v_\beta(P_k) \right] \bar{\xi}_k + o(|\xi_k|)$$

and

$$H_k(\xi_j, \bar{\xi}_j) = \text{const} + \left[ -W(P_k, P_j) + \pi \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} v_\beta(P_k) v_\alpha(P_j) \right] \xi_j +$$

$$\left[ -\pi \sum_{\alpha, \beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} v_\beta(P_k) \bar{v}_\alpha(P_j) \right] \bar{\xi}_j + o(|\xi_j|)$$

for  $j \neq k$  which implies the proposition.  $\square$

### 3 Bergman kernel and holomorphic one-forms.

Let  $B(\cdot, \cdot)$  be the Bergman reproducing kernel for holomorphic differentials (2.7) on  $X$ , let  $P_1, \dots, P_{2g-2}$  be  $2g-2$  distinct points of  $X$  and let  $\xi_k$  be a holomorphic local parameter at  $P_k$ . Introduce the  $(2g-2) \times (2g-2)$  matrix  $\mathcal{B}$ :

$$\mathcal{B} = [B(P_j, P_k)]_{j,k=1,\dots,2g-2},$$

where the value of  $B$  at  $P_k$  is taken with respect to the local parameter  $\xi_k$ . We observe that the rank of  $\mathcal{B}$  is independent of the choice of the local parameter.

According to proposition 1 we have

$$\mathcal{B} = -\frac{1}{\pi} S_{ha}(0).$$

It follows that the  $S$  matrix knows properties of the Riemann surface that are encoded in  $\mathcal{B}$ . For instance we have the following Proposition.

**Proposition 2.** *The following two statements are equivalent*

- (A) *The divisor  $P_1 + \dots + P_{2g-2}$  coincides with the divisor of some holomorphic one-form  $\omega$ .*
- (B)  $\text{rank } \mathcal{B} < g$ .

Before proving this proposition, we recall the

**Definition 1.** *On a Riemann surface  $X$  of genus  $g$ , a divisor  $Q_1 + Q_2 + \dots + Q_g$  is called special when*

$$i(Q_1 + \dots + Q_g) := \dim\{\omega - \text{holomorphic one form} : (\omega) > Q_1 + \dots + Q_g \text{ or } \omega = 0\} > 0.$$

*Equivalently, the divisor  $Q_1 + \dots + Q_g$  is special iff*

$$\det [v_j(Q_k)]_{1 \leq j, k \leq g} = 0. \quad (3.1)$$





**Proposition 3.** *Let*

$$T(\lambda) := S^{ha}(\lambda) = \begin{pmatrix} S^{\xi_1, \bar{\xi}_1}(\lambda) & S^{\xi_1, \bar{\xi}_2}(\lambda) & \dots & S^{\xi_1, \bar{\xi}_{2g-2}}(\lambda) \\ \vdots & \vdots & \dots & \vdots \\ S^{\xi_{2g-2}, \bar{\xi}_1}(\lambda) & S^{\xi_{2g-2}, \bar{\xi}_2}(\lambda) & \dots & S^{\xi_{2g-2}, \bar{\xi}_{2g-2}}(\lambda) \end{pmatrix}.$$

*There exists a constant  $C_g$  that depends only on the genus such that the following comparison formula for the  $\zeta$ -regularized determinants holds:*

$$\det(\Delta_{\text{hol}} - \lambda) = C_g \det T(\lambda) \det(\Delta_F - \lambda).$$

To get Proposition 3 one needs the following reformulation of Proposition 5.3 from [5] for the case of the holomorphic extension.

**Lemma 1.** *The following relation holds true*

$$\text{Tr}((\Delta_{\text{hol}} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}) = -\text{Tr} \left( T(\lambda)^{-1} \frac{d}{d\lambda} T(\lambda) \right). \quad (4.1)$$

*Proof.* As in [5] the key idea is to make use of the M. G. Krein theory for the difference of the resolvents of two self-adjoint extensions of a symmetric operator with finite deficiency indices. Using this theory, for any  $f \in L^2(X)$  there exists a collection  $(x_k)_{k=1, \dots, 2g-2}$  such that

$$(\Delta_{\text{hol}} - \lambda)^{-1} f = (\Delta_F - \lambda)^{-1} f + \sum_{k=1}^{2g-2} x_k G_{\xi_k}(\cdot, \lambda) \quad (4.2)$$

For each  $j$ , considering the coefficient of  $\bar{\xi}_j$  leads to the following equation :

$$\forall j, 0 = \Lambda_{\bar{\xi}_j} ((\Delta_F - \lambda)^{-1} f) + \sum_{k=1}^{2g-2} x_k S^{\xi_k, \bar{\xi}_j}.$$

Since  $S^{\xi_k, \bar{\xi}_j} = T_{kj}$ , by inverting this relation we obtain that

$$\forall k, x_k = - \sum_{j=1}^{2g-2} \Lambda_{\bar{\xi}_j} ((\Delta_F - \lambda)^{-1} f) T_{jk}^{-1}$$

It follows that by taking the trace we obtain

$$\begin{aligned} \text{Tr}((\Delta_{\text{hol}} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}) &= - \sum_{j,k=1}^{2g-2} \sum_{\ell \geq 0} \Lambda_{\bar{\xi}_j} ((\Delta_F - \lambda)^{-1} \phi_\ell) T_{jk}^{-1} \langle \phi_\ell, G_{\xi_k} \rangle \\ &= - \sum_{j,k=1}^{2g-2} T_{jk}^{-1} \sum_{\ell \geq 0} (\lambda_\ell - \lambda)^{-2} \Lambda_{\bar{\xi}_j}(\phi_\ell) \Lambda_{\xi_k}(\phi_\ell) \end{aligned} \quad (4.3)$$

Using equation (2.4), we get

$$\sum_{\ell \geq 0} (\lambda_\ell - \lambda)^{-2} \Lambda_{\bar{\xi}_j}(\phi_\ell) \Lambda_{\xi_k}(\phi_\ell) = \Lambda_{\bar{\xi}_j}(\partial_\lambda G_{\xi_k}) = \partial_\lambda S^{\xi_k, \bar{\xi}_j} = \partial_\lambda T_{kj}.$$

The Lemma thus follows.  $\square$

This lemma implies Proposition 3 by performing exactly the same contour integration as in [5] (from that point, the remaining part of the proof can be repeated verbatim).

## 4.2 DtN isospectrality of $\Delta_F$ and $\Delta_{\text{hol}}$

Let the divisor  $P_1 + \dots + P_{2g-2}$  be in the canonical class. Then the corresponding Troyanov metric (all the conical angles are equal to  $4\pi$ ) has trivial holonomy and is given by  $|\omega|^2$ , where  $\omega$  is a holomorphic one-form with simple zeros at  $P_1, \dots, P_{2g-2}$ . Let  $z(P) = \int_{P_0}^P \omega$  with some base point  $P_0$ . Then  $z(P)$  can be taken as holomorphic local parameter on  $X$  in a vicinity of any point of  $X \setminus \{P_1, \dots, P_{2g-2}\}$ . Consider the operator

$$D_z : L^2(X, |\omega|^2) \supset C_0^\infty(X \setminus \{P_1, \dots, P_{2g-2}\}) \ni u \rightarrow u_z \in L^2(X, |\omega|^2)$$

**Remark 3.** Let  $\partial_z$  be the standard Cauchy-Riemann operator  $\partial_z : C^\infty(X) \rightarrow \Lambda^{1,0}(X)$ . Then  $D_z = \frac{1}{\omega} \partial_z$ .

The operator  $D_z$  is closable. Denote its closure again by  $D_z$ . The following observation belongs to the first author [4].

**Proposition 4.** One has the following identifications of the self-adjoint operators

$$D_z^* D_z = \frac{1}{4} \Delta_F,$$

$$D_z D_z^* = \frac{1}{4} \Delta_{\text{hol}}.$$

*Proof.* By general considerations, both operators are self-adjoint extensions of the Laplace operator on functions that vanish near the conical points. It thus suffices to prove that the domains coincide. For the first operator, it follows by remarking that  $\text{dom}(D_z^* D_z) \subset \text{dom}(D_z)$ . For the second operator it follows by remarking that any function in  $\text{dom}(\Delta^*)$  with holomorphic behaviour near the conical points is in  $\text{dom}(D_z D_z^*)$ .  $\square$

From Proposition 4 it immediately follows that the operators  $\Delta_F$  and  $\Delta_N$  are DtN isospectral, that is

$$\dim \text{Ker}(\Delta_F - \lambda) = \dim \text{Ker}(\Delta_{\text{hol}} - \lambda)$$

for any  $\lambda \neq 0$ .

To compare the spectral determinants of both extensions, it thus suffices to compute the dimension of their kernels. The kernel of  $\Delta_F$  only consists of the constant functions.

**Lemma 1.** For metrics with trivial holonomy one has the equality

$$\dim \text{Ker} \Delta_{\text{hol}} = g.$$

*Proof.* Denote by  $X_\epsilon$  the surface  $X$  where  $\epsilon$ -disks around  $P_k$  have been deleted. Then, for any  $u \in \text{dom}(\Delta_{\text{hol}})$  we have .

$$\begin{aligned} -\frac{1}{4} \langle u, \Delta u \rangle &= \lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} \bar{u} \partial_z \partial_{\bar{z}} u |dz|^2 \\ &= \langle \partial_{\bar{z}} u, \partial_{\bar{z}} u \rangle \\ &\quad + \lim_{\epsilon \rightarrow 0} \sum_{k=1}^{2g-2} \oint_{|\xi_k|=\sqrt{\epsilon}} (A_k/\xi_k + \dots) \frac{1}{\xi_k} \partial_{\bar{\xi}_k} (A_k/\xi_k + B + C\xi_k + O(|\xi_k|^2)) \bar{\xi}_k d\bar{\xi}_k \\ &= \langle \partial_{\bar{z}} u, \partial_{\bar{z}} u \rangle. \end{aligned}$$

It follows that if  $u \in \text{Ker } \Delta_{\text{hol}}$  then  $\partial_{\bar{z}}u = 0$ . Thus, the one form  $u\omega$  is holomorphic and, therefore,  $u \in \text{Span}\{v_1/\omega, \dots, v_g/\omega\}$ . Conversely, for any holomorphic one-form  $v$  the meromorphic function  $v/\omega$  is seen to be in  $\text{dom}(\Delta_{\text{hol}})$  and satisfies  $\Delta^*v/\omega = 0$ . We obtain

$$\ker(\Delta_{\text{hol}}) = \text{Span}\{v_1/\omega, \dots, v_g/\omega\}.$$

□

**Remark 4.** *The needed equality also follows from the DtN isospectrality of  $\Delta_{\text{hol}}$  and  $\Delta_F$  and Remark 5.11 from [5]. Indeed, the latter implies the relation*

$$\zeta(0, \Delta_{\text{hol}} - \lambda) = \zeta(0, \Delta_F - \lambda) + g - 1$$

between the values of the operator zeta-functions of  $\Delta_{\text{hol}} - \lambda$  and  $\Delta_F - \lambda$  at  $s = 0$ .

The DtN isospectrality of  $\Delta_{\text{hol}}$  and  $\Delta_F$  (see [4], Theorem 4.8) then implies

$$\det(\Delta_{\text{hol}} - \lambda) = \lambda^{g-1} \det(\Delta_F - \lambda)$$

and, therefore,

$$\frac{\det T(\lambda)}{\lambda^{g-1}} = C_g \tag{4.4}$$

and

$$C_g = \frac{1}{(g-1)!} \left( \frac{d}{d\lambda} \right)^{g-1} \det T(\lambda) \Big|_{\lambda=0}, \tag{4.5}$$

For any integer  $M$ , denote by  $\mathcal{N}_M$  the set of  $(2g-2)$ -tuples  $\mathbf{n} := (n_1, \dots, n_{2g-2})$  of non-negative integers such that  $n_1 + \dots + n_{2g-2} = M$ . The preceding identity can thus be reformulated as

$$C_g = \sum_{\mathbf{n} \in \mathcal{N}(g-1)} \frac{1}{n_1! \dots n_{2g-2}!} \left| \begin{array}{ccc} \left( \frac{d}{d\lambda} \right)^{n_1} S^{\xi_1 \bar{\xi}_1}(0) & \dots & \left( \frac{d}{d\lambda} \right)^{n_{2g-2}} S^{\xi_1 \bar{\xi}_{2g-2}}(0) \\ \vdots & \dots & \vdots \\ \left( \frac{d}{d\lambda} \right)^{n_1} S^{\xi_{2g-2} \bar{\xi}_1}(0) & \dots & \left( \frac{d}{d\lambda} \right)^{n_{2g-2}} S^{\xi_{2g-2} \bar{\xi}_{2g-2}}(0) \end{array} \right| \tag{4.6}$$

and  $C_g$  is an absolute constants that depends on the genus only.

We now give alternative expressions for the coefficients in the preceding determinant.

**Lemma 2.** *For all  $n \geq 1$  and all  $j, k$ , the following generalization of the formula (4.2) from [6] holds:*

$$(\partial_\lambda)^n S^{\xi_k \bar{\xi}_j}(\lambda) = n! \int_X [(\Delta_F - \lambda)^{1-n} G_{\xi_k}] G_{\bar{\xi}_j} dS. \tag{4.7}$$

*Proof.* We start from (2.4), apply  $\Lambda_{\bar{\xi}_j}$  and differentiate  $n-1$  times to obtain

$$\begin{aligned} (\partial_\lambda)^n S^{\xi_k \bar{\xi}_j}(\lambda) &= n! \sum_{\ell \geq 0} (\lambda_\ell - \lambda)^{-1-n} \Lambda_{\xi_k}(\phi_\ell) \Lambda_{\bar{\xi}_j}(\phi_\ell) \\ &= n! \sum_{\ell \geq 0} (\lambda_\ell - \lambda)^{1-n} \frac{\Lambda_{\xi_k}(\phi_\ell)}{\lambda_\ell - \lambda} \frac{\Lambda_{\bar{\xi}_j}(\phi_\ell)}{\lambda_\ell - \lambda} \\ &= n! \int_X [(\Delta_F - \lambda)^{1-n} G_{\xi_k}] G_{\bar{\xi}_j} dS. \end{aligned}$$

□

From the expansion in the eigenfunctions basis, we now observe that  $G_{\xi_k} \perp 1$  and is analytic at 0. It follows that

$$G_{\xi_k}(\cdot; \lambda = 0) = \mathcal{H}_k := \frac{1}{2\sqrt{\pi}} \left( H_k - \frac{1}{A} \int_X H_k \right), \quad (4.8)$$

where  $H_k$  is explicitly given by (2.8).

The operator

$$\Delta_F^{-1} : 1^\perp \rightarrow 1^\perp$$

has an integral kernel which is given by the Green function

$$G(x, y) = \frac{1}{2\pi A^2} \int_X \int_X \Re \left( \int_b^x \Omega_{y-a} \right) dS(a) dS(b) \quad (4.9)$$

where

$$\Omega_{a-b}(z) = \int_a^b W(z, \cdot) - 2\pi i \sum_{\alpha\beta} (\Im \mathbb{B})_{\alpha\beta}^{-1} v_\alpha(z) \Im \int_a^b v_\beta$$

(see, [3], formula (2.19)). Thus, by letting  $\lambda$  go to 0 we obtain the following explicit expression for all the terms in the r. h. s. of (4.6) :

$$\left( \frac{d}{d\lambda} \right)_{|\lambda=0}^{n_k} S^{\xi_l \bar{\xi}_k}(\lambda) = \int_X [(\Delta_F|_{1^\perp})^{1-n_k} \mathcal{H}_l] \overline{\mathcal{H}_k} \quad (4.10)$$

where

$$(\Delta_F|_{1^\perp})^{1-n_k} \mathcal{H}_l(x) = \quad (4.11)$$

$$\int_X G(x, x_1) \int_X G(x_1, x_2) \dots \int_X G(x_{n_k-1}, y) \mathcal{H}_l(y) dS(y) dS(x_{n_k-1}) \dots dS(x_1).$$

## 5 Metrics with nontrivial holonomy

**Lemma 2.** *Let  $P_1, \dots, P_{2g-2}$  be points of  $X$  such that the divisor  $D = P_1 + \dots + P_{2g-2}$  is not canonical. (According to Proposition 2, this condition is equivalent to the condition  $\text{rank } \mathcal{B} = g$ .) Then we have*

$$\dim \text{Ker } \Delta_{\text{hol}} = g - 1.$$

*Proof.* Let  $u \in \text{Ker } \Delta_{\text{hol}}$ . Then, similarly to the proof of Lemma 1, one can show that  $u$  is holomorphic in  $X \setminus \{P_1, \dots, P_{2g-2}\}$ . According to Corollary of Theorem III.9.10 from [2],  $D$  is the divisor of a holomorphic multi-valued Prym differential  $\omega$  with nontrivial (multiplicative) unitary monodromy  $T_{a_\alpha}, T_{b_\alpha}$  along basic cycles  $a_\alpha, b_\alpha$ :

$$T_{a_\alpha} \omega = e^{ir_\alpha} \omega; T_{b_\alpha} \omega = e^{is_\alpha} \omega; \quad r_\alpha, s_\alpha \in \mathbb{R}.$$

Then  $u\omega$  is a Prym differential with the same monodromy. Since the space of Prym differentials with a given nontrivial unitary monodromy has dimension  $g - 1$  (see [10], p. 147), one gets

$$\dim \text{ker } \Delta_{\text{hol}} \leq g - 1.$$

The inequality

$$\dim \text{ker } \Delta_{\text{hol}} \geq g - 1$$

is obvious: take  $\{\omega_1, \dots, \omega_{g-1}\}$  a basis of the space of Prym differentials with the same monodromy as  $\omega$ , then the functions  $\omega_1/\omega, \dots, \omega_{g-1}/\omega$  are linearly independent, belong to  $\text{dom}(\Delta_{\text{hol}}) \cap \ker \Delta^*$ .  $\square$

Lemma 2 and Proposition 3 imply the following Theorem.

**Theorem 2.** *Let the divisor  $P_1 + \dots + P_{2g-2}$  do not belong to the canonical class. Then*

$$\frac{\det \Delta_{\text{hol}}}{\det \Delta_F} = c_g \cdot \sum_{\mathbf{n} \in \mathcal{N}(g-2)} \frac{1}{n_1! \dots n_{2g-2}!} \begin{vmatrix} \left(\frac{d}{d\lambda}\right)^{n_1} S^{\xi_1 \bar{\xi}_1}(0) & \dots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}} S^{\xi_1 \bar{\xi}_{2g-2}}(0) \\ \vdots & \dots & \vdots \\ \left(\frac{d}{d\lambda}\right)^{n_1} S^{\xi_{2g-2} \bar{\xi}_1}(0) & \dots & \left(\frac{d}{d\lambda}\right)^{n_{2g-2}} S^{\xi_{2g-2} \bar{\xi}_{2g-2}}(0) \end{vmatrix},$$

where all the terms at the right are explicitly given by (4.8, 4.9, 4.10, 4.11) and the absolute constant  $c_g$  depends only on the genus  $g$ . In particular, in genus 2 one gets relation (1.3).

## References

- [1] Egan F., Symmetric matrices and quadratic forms, The Mathematical Gazette, Vol. 29, N. 285 (Jul., 1945), pp. 89–91
- [2] Farkas H., Kra I., Riemann surfaces, Springer, 1980
- [3] Fay J., Kernel functions, analytic torsion and moduli spaces, Memoirs of AMS, vol. 92 (1992)
- [4] L. Hillairet, Spectral theory of translation surfaces: a short introduction, Institut Fourier: Actes de séminaire de Théorie spectrale et géométrie. Vol. 28 (2009–2010), p. 51–62
- [5] L. Hillairet, A. Kokotov, Krein formula and  $S$ -matrix for Euclidean surfaces with conical singularities, Journal of Geometric Analysis, Vol. 23 (2013), N3, 1498–1529
- [6] L. Hillairet, V. Kalvin and A. Kokotov, Moduli spaces of meromorphic functions and determinant of Laplacian, arXiv:1410.3106 [math.SP]
- [7] A. Kokotov, Polyhedral surfaces and determinant of Laplacian, Proc. Amer. Math. Soc. 141 (2013), 725–735
- [8] A. Kokotov, D. Korotkin, Tau-functions on spaces of Abelian differentials and higher genus generalization of Ray-Singer formula, Journal of Differential Geometry, Vol. 82, n. 1, (2009), 35–100
- [9] Troyanov M., Les surfaces euclidiennes à singularités coniques, Enseign. Math., 32 (1-2), 79–94 (1986)
- [10] Weyl H., The concept of a Riemann surface, Addison Wesley Publishing Company, Reading, Massachusetts, 1955.